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Investigation of the static and dynamic behaviour of anisotropic cylindrical bodies with noncircular cross-section

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Abstract

A general approach to the solution of some static and dynamic problems for hollow anisotropic cylinders with noncircular cross-section within the framework of the three-dimensional linear elasticity theory is proposed. The approach is based on a steady numerical method and is realized in a program complex. The stress–strain state and certain types of dynamic characteristics of hollow anisotropic cylinders with noncircular cross-section are studied with their geometric and mechanic parameters being varied over a wide range. The effect of the cylinder cross-section shape on the character of mechanical behaviour is examined.

Keywords: Noncircular cylinders; Numerical approach; Anisotropic material; Statics; Dynamics

1. Introduction

Noncircular hollow cylinders, as structural elements, are widely used in constructing the high strength and reliable structures, acted upon by nonuniform loads, in such fields as engineering and mining industries as well as in other branches of the modern technique. To ensure the strength and reliability of the structures it is necessary to know the stress–strain state and dynamical characteristics of the structural elements making these structures.

There exists an abundant literature on the solution of the problems on the stress–strain state, stability, and vibrations of thin-walled noncircular cylindrical bodies. These problems usually are solved within the framework of the classical and various refined theories of shells (Cheung et al., 1991; Mc Daniel and Logan, 1971; Meyers and Hyer, 1977; Noor, 1973; Noor and Burton, 1992; Soldatos, 1985, 1986). However, in many cases the geometrical sizes of structural elements and pronounced anisotropy of a material make it necessary to use the three-dimensional theory of elasticity.

The solution of problems concerning the statics and dynamics of thick-walled elements as spatial problems of elasticity theory, is faced with significant difficulties attributed to complexity of the initial

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system of partial differential equations and to necessity to satisfy the boundary conditions on the surfaces bounding the given elastic body. These difficulties rise substantially upon the calculation of such structural elements as cylinders of complex shape made of anisotropic and inhomogeneous materials. For this reason there is only a sparse number of studies on the subject (Guz, 1986; Soldatos, 1999).

Therefore, of great importance is the development of efficient approaches to the solution of stress–strain problems and to the determination of dynamic characteristics of hollow cylinders in the three-dimensional statement, which allow us, using modern personal computers, to solve exactly the problems mentioned with the geometric and mechanical parameters being varied over a wide range.

Recently, the development of computational technique has made it possible to solve a large number of problems of the elasticity theory by using the approximate methods such as the method of straight lines, method of integral relations, finite-difference method, and method of finite elements that are based on discretization of the exact equations of the spatial elasticity theory. The study of the initial problem, employing these methods, has been reduced to the solution of the system of ordinary differential equations or of the system of algebraic equations. Apart from the finite element method, which is widely used today in solving the spatial problems of the elasticity theory (Tahbildar and Gladwell, 1972), the other approaches are applied (Hutchinson and El-Arhari, 1986; Grigorenko et al., 2000).

In the present work, the authors propose a general efficient numerical–analytical approach for investigation of the stress–strain state and some dynamical characteristics of anisotropic hollow thick-walled cylinders with a noncircular cross-section under specified boundary conditions at their bounding surfaces and ends. The approach is based on the reduction of the initial equations of spatial elasticity theory to systems of ordinary differential equations for boundary-value problems and for problems on eigenvalues. All the parameters, characterizing a stress–strain state and external loads (for static problems), are expanded into the Fourier series in a longitudinal coordinate. For dynamic problems, the solutions are represented as a traveling wave. In both cases, the difference approximation across the cylinder thickness is used. One-dimensional problems are solved using the exact stable numerical method of discrete orthogonalization (Bellman and Kalaba, 1965; Grigorenko and Rozhok, 2003), and dynamical problems are considered within the framework of the same method combined with the procedure of stepwise search. So, the results obtained are sufficiently high accuracy along the longitudinal and arc coordinates due to use of continuum-based methods. The discrete finite difference method being distinguished from the finite element method is used only in one direction along the cylinder thickness (for sufficiently smooth area). In using the finite element method the solution is discretely approximated along all of three directions. The approach developed is realized with the help of the computational complex including modern personal computers. Main positions of the approach are given in the authors' works (Grigorenko and Vlaikov, 1988; Grigorenko, 1997; Vlaikov and Shevchenko, 1988).

2. Simulation: basic equations

In the present work an elastic body in the form of hollow noncircular cylinder has been chosen, as the subject of consideration. In solving the problem we will use the orthogonal curvilinear coordinate system $\alpha = s$, $\beta = t$, and γ (Fig. 1), where $s = \text{const.}$, $t = \text{const.}$ are the lines of principal curvatures on some coordinate cylindrical surface that represent the families of directrices and generatrices, with the coordinate γ being counted off along the normal to this surface. We will count off the arc coordinate t of the directrix from the certain fixed generatrix and set the natural parameter s to be equal to the distance from one of the boundary contours of a coordinate surface (the boundary contours are chosen from the family of directing curves that is formed by the cross-sections of the coordinate surface of the cylinder). Thus, the position of some point M in a space is uniquely defined by three parameters (s, t, γ).

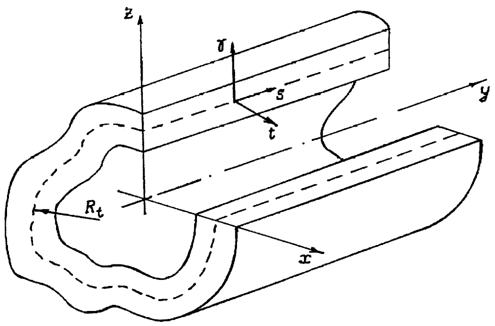

Fig. 1. Coordinate system adopted.

As the coordinate surface of the cylinders under study, we may take any surface that is equidistant to the surfaces bounding the cylinder. In practical calculations, a middle surface, which is equidistant from lateral ones, is usually taken as the coordinate surface.

In accepted coordinate system, the squared length of a linear element of the cylinder is

$$ds^2 = H_1^2\, ds^2 + H_2^2\, dt^2 + H_3^2\, d\gamma^2, \tag{1}$$

where $H_1 = 1$, $H_2 = 1 + \frac{\gamma}{R_t}$, $H_3 = 1$, and R_t is the radius of curvature of the directrix of the middle surface.

Let us refer the plane of the cylinder cross-section to the Cartesian coordinate system x, z. In parametric form the equations for the directrix of the middle surface can be represented as

$$x = x(t), \quad z = z(t).$$

Then the radius of curvature of the directrix is defined as

$$R_t = \sqrt{\frac{(\dot{x}^2 + \dot{y}^2)^3}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}}. \tag{2}$$

Thus, in order to specify the geometry of a cylinder, it is necessary to set the geometry of its middle surface and its thickness.

Upon the deformation of a hollow noncircular cylinder referred to the curvilinear orthogonal system s, t, γ, all its points will occupy new positions in a space. Then the full displacement of some point of the cylinder can be presented by three components

$$\begin{aligned} u_s &= u_s(s, t, \gamma), \\ u_t &= u_t(s, t, \gamma), \\ u_\gamma &= u_\gamma(s, t, \gamma), \end{aligned} \tag{3}$$

which are the projections of the full displacement of the point on the directions of tangents to the coordinate lines s, t, γ. In what follows, we will call the quantities u_s, u_t, u_γ as displacements along the generatrix, directrix, and in the normal direction, respectively. Since the cylinder is considered as a continuous elastic body, then u_s, u_t, u_γ must be continuous functions of the variables s, t, γ over the whole volume.

Displacements and strains are connected by the following relations:

$$
\begin{aligned}
e_s &= \frac{\partial u_s}{\partial s},\\
e_t &= \frac{1}{H_2}\frac{\partial u_t}{\partial t} + \frac{1}{H_2}\frac{\partial H_2}{\partial \gamma}u_\gamma,\\
e_\gamma &= \frac{\partial u_\gamma}{\partial \gamma},\\
e_{st} &= \frac{1}{H_2}\frac{\partial u_s}{\partial t} + \frac{\partial u_t}{\partial s}\\
e_{t\gamma} &= \frac{\partial u_t}{\partial t} - \frac{1}{H_2}\frac{\partial H_2}{\partial \gamma}u_t + \frac{1}{H_2}\frac{\partial u_\gamma}{\partial t},\\
e_{s\gamma} &= \frac{\partial u_\gamma}{\partial s} + \frac{\partial u_s}{\partial \gamma},
\end{aligned}
\tag{4}
$$

where e_s, e_t, e_γ are the linear strains along the directions of the coordinate lines and e_{st}, $e_{t\gamma}$, $e_{s\gamma}$ are the shear strains. These relations enable us to determine strains by the known displacements that are held in the body.

The conditions of equilibrium can be represented as

$$
\begin{aligned}
&H_2\frac{\partial \sigma_s}{\partial s} + \frac{\partial \tau_{st}}{\partial t} + \frac{\partial}{\partial \gamma}(H_2\tau_{s\gamma}) + H_2P_s = 0,\\
&\frac{\partial \sigma_t}{\partial t} + H_2\frac{\partial \tau_{st}}{\partial s} + 2\frac{\partial H_2}{\partial \gamma}\tau_{t\gamma} + H_2\frac{\partial \tau_{t\gamma}}{\partial \gamma} + H_2P_t = 0,\\
&\frac{\partial}{\partial \gamma}(H_2\sigma_\gamma) - \frac{\partial H_2}{\partial \gamma}\sigma_t + H_2\frac{\partial \tau_{s\gamma}}{\partial s} + \frac{\partial \tau_{t\gamma}}{\partial t} + H_2P_\gamma = 0.
\end{aligned}
\tag{5}
$$

Here, σ_s, σ_t, and σ_γ are the normal stresses, τ_{st}, $\tau_{t\gamma}$, and $\tau_{\gamma s}$ are the shear stresses, and $P_s(s,t,\gamma)$, $P_t(s,t,\gamma)$, and $P_\gamma(s,t,\gamma)$ are the projections of volume forces on the corresponding directions of tangents to the coordinate lines.

The equations of motion can be written in the following form:

$$
\begin{aligned}
&H_2\frac{\partial \sigma_s}{\partial s} + \frac{\partial \tau_{st}}{\partial t} + \frac{\partial}{\partial \gamma}(H_2\tau_{s\gamma}) - \rho\frac{\partial^2 u_s}{\partial t_*^2} = 0,\\
&\frac{\partial \sigma_t}{\partial t} + H_2\frac{\partial \tau_{st}}{\partial s} + 2\frac{\partial H_2}{\partial \gamma}\tau_{t\gamma} + H_2\frac{\partial \tau_{t\gamma}}{\partial \gamma} - \rho\frac{\partial^2 u_t}{\partial t_*^2} = 0,\\
&\frac{\partial}{\partial \gamma}(H_2\sigma_\gamma) - \frac{\partial H_2}{\partial \gamma}\sigma_t + H_2\frac{\partial \tau_{s\gamma}}{\partial s} + \frac{\partial \tau_{t\gamma}}{\partial t} - \rho\frac{\partial^2 u_\gamma}{\partial t_*^2} = 0,
\end{aligned}
\tag{6}
$$

where t_* stands for the temporal coordinate.

For an orthotropic material, whose symmetry planes of elastic properties are orthogonal to the coordinate lines, the relations of the generalized Hooke law can be written as

$$
\begin{aligned}
\sigma_s &= c_{11}e_s + c_{12}e_t + c_{13}e_\gamma,\\
\sigma_t &= c_{21}e_s + c_{22}e_t + c_{23}e_\gamma,\\
\sigma_\gamma &= c_{31}e_s + c_{32}e_t + c_{33}e_\gamma,\\
\tau_{t\gamma} &= c_{44}e_{t\gamma}, \quad \tau_{\gamma s} = c_{55}e_{\gamma s}, \quad \tau_{st} = c_{66}e_{st}.
\end{aligned}
\tag{7}
$$

Here

$$
\begin{aligned}
&c_{11} = \frac{1}{\Omega}(g_{22}g_{33} - g_{23}^2), \quad c_{12} = c_{21} = \frac{1}{\Omega}(g_{31}g_{23} - g_{21}g_{33}),\\
&c_{13} = c_{31} = \frac{1}{\Omega}(g_{21}g_{32} - g_{31}g_{22}), \quad c_{22} = \frac{1}{\Omega}(g_{11}g_{33} - g_{43}^2),\\
&c_{23} = c_{32} = \frac{1}{\Omega}(g_{12}g_{31} - g_{11}g_{32}), \quad c_{33} = \frac{1}{\Omega}(g_{11}g_{22} - g_{12}^2),\\
&c_{44} = \frac{1}{g_{44}}, \quad c_{55} = \frac{1}{g_{55}}, \quad c_{66} = \frac{1}{g_{66}},\\
&\Omega = (g_{22}g_{33} - g_{23}^2)g_{11} + (g_{31}g_{23} - g_{21}g_{33})g_{12} + (g_{21}g_{32} - g_{31}g_{22})g_{13},
\end{aligned} \tag{8}
$$

where the coefficients g_{lk} are related to mechanical characteristics as follows (Lekhnitsky, 1977):

$$
\begin{aligned}
&g_{11} = \frac{1}{E_s}, \quad g_{22} = \frac{1}{E_t}, \quad g_{33} = \frac{1}{E_\gamma},\\
&g_{12} = -\frac{v_{st}}{E_t}, \quad g_{23} = -\frac{v_{t\gamma}}{E_\gamma}, \quad g_{13} = -\frac{v_{\gamma s}}{E_s},\\
&g_{44} = \frac{1}{G_{t\gamma}}, \quad g_{55} = \frac{1}{G_{\gamma s}}, \quad g_{66} = \frac{1}{G_{st}}.
\end{aligned} \tag{9}
$$

In (7)–(9), E_s, E_t, E_γ are the moduli of elasticity along the directions s, t, γ, respectively; G_{st}, $G_{t\gamma}$, $G_{\gamma s}$ are the shear moduli for the planes parallel to the coordinate surfaces $\alpha = \text{const.}$, $s = \text{const.}$, $\gamma = \text{const.}$; v_{st}, $v_{t\gamma}$, $v_{s\gamma}$ are Poisson's ratios characterizing the transverse contraction under tension in the directions of the coordinate axes.

Let us impose some restrictions on the elastic constants introduced in the relations of the generalized Hooke law. The sum of the works produced by all the stress components should be positive. Such condition restricts the value of elastic constants and reflects the fact that the potential energy is a positive definite quadratic form (Grigorenko et al., 1977).

Thus, relations (4)–(7) are the basic equations of the elasticity theory for noncircular hollow cylinders. It should be noted that in solving the problems of the elasticity theory for noncircular hollow cylinders, it is necessary to satisfy not only the main equations presented above, but also the boundary conditions.

Such conditions for inner and outer surfaces of the cylinder are written as follows:

for static problems

$$
\begin{aligned}
&\sigma_\gamma^0 = q_\gamma^-, \quad \sigma_{s\gamma}^0 = q_s^-, \quad \tau_{t\gamma}^0 = q_t^- \qquad \text{at } \gamma = -H/2;\\
&\sigma_\gamma^N = q_\gamma^+, \quad \tau_{s\gamma}^N = q_s^+, \quad \tau_{t\gamma}^N = q_t^+ \qquad \text{at } \gamma = +H/2;
\end{aligned} \tag{10}
$$

for dynamical problems

$$
\begin{aligned}
&\sigma_\gamma^0 = 0, \quad \sigma_{s\gamma}^0 = 0, \quad \tau_{t\gamma}^0 = 0 \qquad \text{at } \gamma = -H/2;\\
&\sigma_\gamma^N = 0, \quad \tau_{s\gamma}^N = 0, \quad \tau_{t\gamma}^N = 0 \qquad \text{at } \gamma = +H/2.
\end{aligned} \tag{11}
$$

The conditions (10) and (11) may be substituted by the conditions given in terms of displacements or in mixed form.

In addition to the conditions on the bounding surfaces $\gamma = -H/2$ and $\gamma = H/2$, we should satisfy the conditions on the cylinder ends, $s = 0$ and $s = \ell$, and in the cross-sections $t = t_0$ and $t = t_p$. Here, ℓ is the cylinder length, $t_0 = \text{const.}$ and $t_p = \text{const.}$ are the cross-sections limiting the cylinder along the directrix.

Boundary conditions can be formulated in terms of stresses, displacements, or in mixed form through the functions σ_s, τ_{st}, $\tau_{s\gamma}$, u_s, u_t, and u_γ on the cylinder ends and σ_t, τ_{st}, $\tau_{t\gamma}$, u_s, u_t, and u_γ in the cross-sections limiting the cylinder along the directrix.

The solution of the problem for closed cylinders is reduced to that for nonclosed ones satisfying the conditions of periodicity

$$R(s, t_0, \gamma) = R(s, t_p, \gamma), \tag{12}$$

where

$$R(s, t, \gamma) = \{\sigma_t, \tau_{st}, \tau_{t\gamma}, u_s, u_t, u_\gamma\}.$$

3. Solution of partial differential equations for an anisotropic noncircular cylinder

In constructing the solving system of differential equations that describe the stress–strain state of a noncircular hollow cylinder and the associated steady dynamic processes, we use the above presented equations of the spatial elasticity theory which include three differential equations of equilibrium (5), three equations of motion (6), six Cauchy relations (4), and six relations of the generalized Hooke law (7).

To solve the problem we adopt that the directrix of the middle surface is an arbitrary continuous piecewise smooth curve, the mechanical characteristics vary along the directrix and over the thickness, being constant along the generatrix, and the law of distribution of the surface and volume loads acting on the cylinder can be given in various ways. In the case of dynamic problems, it is supposed that the lateral surfaces of the cylinder are free from stresses.

As solving functions, we have chosen three components of stresses σ_t, τ_{st}, $\tau_{\gamma t}$ and three components of displacements u_s, u_t, u_γ. Such a choice of solving functions is caused by the fact that the boundary conditions at the cross-sections limiting the cylinder along the directrix are formulated just in terms of these components.

Thus, the required system of differential equations should be solved for partial derivatives of the components of the chosen functions with respect to the coordinate t, and the right-hand sides can include, besides the factors of an external load, only the solving functions and their derivatives with respect to the coordinates s and γ.

In the relations of the generalized Hooke law that express stresses σ_t, τ_{st}, $\tau_{\gamma t}$ in terms of strains, we exclude the appropriate components of deformation with the help of the Cauchy relations and solve the first equation for $\frac{\partial u_t}{\partial t}$, the second one for $\frac{\partial u_s}{\partial t}$, and the third one for $\frac{\partial u_\gamma}{\partial t}$. Thus, we immediately obtain relations expressing the derivatives $\frac{\partial u_s}{\partial t}, \frac{\partial u_t}{\partial t}, \frac{\partial u_\gamma}{\partial t}$ in terms of the solving functions and their derivatives with respect to the coordinates s and γ:

$$\begin{aligned}
\frac{\partial u_s}{\partial t} &= \frac{H_2}{c_{66}}\tau_{st} - H_2\frac{\partial u_t}{\partial s},\\
\frac{\partial u_t}{\partial t} &= \frac{H_2}{c_{22}}\sigma_t - \frac{H_2}{c_{22}}c_{21}\frac{\partial u_s}{\partial s} - \frac{\partial H_2}{\partial \gamma}u_\gamma - \frac{H_2}{c_{22}}c_{23}\frac{\partial u_\gamma}{\partial \gamma},\\
\frac{\partial u_\gamma}{\partial t} &= \frac{H_2}{c_{44}}\tau_{t\gamma} + \frac{\partial H_2}{\partial \gamma}u_t - H_2\frac{\partial u_t}{\partial \gamma}.
\end{aligned} \tag{13}$$

In three remaining equations of the generalized Hooke law, we have also excluded deformations with the help of the Cauchy relations. The formulas obtained for σ_s and σ_γ include the derivatives $\frac{\partial u_t}{\partial t}$ that can be excluded by using the second equation in (13). After transformations, the formulas for σ_s, σ_γ, $\tau_{s\gamma}$ can be represented in the following form:

$$
\begin{aligned}
&\sigma_s = M_{11}\sigma_t + M_{12}\frac{\partial u_s}{\partial s} + M_{13}\frac{\partial u_\gamma}{\partial \gamma},\\
&\sigma_\gamma = M_{21}\sigma_t + M_{22}\frac{\partial u_s}{\partial s} + M_{23}\frac{\partial u_\gamma}{\partial \gamma},\\
&\tau_{s\gamma} = M_{31}.
\end{aligned}
\tag{14}
$$

Here, M_{ij} are the coefficients that can be expressed in terms of the physical and mechanical characteristics of the cylinder as

$$
\begin{aligned}
&M_{11} = \frac{c_{12}}{c_{22}}, \quad M_{12} = c_{11} - \frac{c_{12}^2}{c_{22}},\\
&M_{13} = c_{13} - \frac{c_{12}}{c_{22}}c_{23},\\
&M_{21} = \frac{c_{32}}{c_{22}}, \quad M_{22} = c_{31} - \frac{c_{32}}{c_{22}}c_{21},\\
&M_{23} = c_{33} - \frac{c_{23}^2}{c_{22}},\\
&M_{31} = c_{55}, \quad M_{32} = c_{55}.
\end{aligned}
\tag{15}
$$

By differentiating the formulas for σ_γ and $\tau_{s\gamma}$ with respect to the coordinate γ and regarding for the inhomogeneity of the material over the thickness, we get the formulas for the derivatives $\frac{\partial \sigma_\gamma}{\partial \gamma}$ and $\frac{\partial \tau_{s\gamma}}{\partial \gamma}$ in the following form:

$$
\begin{aligned}
&\frac{\partial \sigma_\gamma}{\partial \gamma} = \frac{\partial M_{21}}{\partial \gamma}\sigma_t + M_{21}\frac{\partial \sigma_t}{\partial \gamma} + \frac{\partial M_{22}}{\partial \gamma}\frac{\partial u_s}{\partial s} + M_{22}\frac{\partial^2 u_s}{\partial s \partial \gamma} + \frac{\partial M_{23}}{\partial \gamma}\frac{\partial u_\gamma}{\partial \gamma} + M_{23}\frac{\partial^2 u_\gamma}{\partial \gamma^2},\\
&\frac{\partial \tau_{s\gamma}}{\partial \gamma} = \frac{\partial M_{31}}{\partial \gamma}\frac{\partial u_\gamma}{\partial s} + M_{31}\frac{\partial^2 u_\gamma}{\partial s \partial \gamma} + \frac{\partial M_{32}}{\partial \gamma}\frac{\partial u_s}{\partial \gamma} + M_{32}\frac{\partial^2 u_s}{\partial \gamma^2}.
\end{aligned}
\tag{16}
$$

In what follows, relations (14) and (16) are used in derivating the system of solving equations and are applied for the calculation of all the components of a stress state.

First of all, we solve the second equation of equilibrium in (5) for $\frac{\partial \sigma_t}{\partial t}$. The first equation of equilibrium is solved for $\frac{\partial \tau_{st}}{\partial t}$ and $\frac{\partial \sigma_s}{\partial s}$, with $\tau_{s\gamma}$ and $\frac{\partial \tau_{s\gamma}}{\partial \gamma}$ being excluded with the help of the relations obtained above. Then we solve the third equilibrium equation for $\frac{\partial \tau_{t\gamma}}{\partial t}$ and exclude both $\frac{\partial \tau_{s\gamma}}{\partial s}$ and $\frac{\partial \sigma_\gamma}{\partial \gamma}$.

The relations obtained compose the system of six differential equations that are solved relative to the partial derivatives of the functions σ_t, τ_{st}, $\tau_{t\gamma}$, u_s, u_t, and u_γ with respect to the coordinate t and contain these functions or their derivatives with respect to the coordinates s and γ on the right-hand sides. They are written as follows:

$$
\begin{aligned}
&\frac{\partial \sigma_t}{\partial t} = L_{11}\frac{\partial \tau_{st}}{\partial s} + L_{12}\tau_{t\gamma} + L_{13}\frac{\partial \tau_{t\gamma}}{\partial \gamma} + L_{14}P_t;\\
&\frac{\partial \tau_{st}}{\partial t} = L_{21}\frac{\partial \sigma_t}{\partial s} + L_{22}\frac{\partial^2 u_s}{\partial s^2} + L_{23}\frac{\partial u_\gamma}{\partial s} + L_{24}\frac{\partial u_s}{\partial \gamma} + L_{25}\frac{\partial^2 u_\gamma}{\partial s \partial \gamma} + L_{26}\frac{\partial^2 u_s}{\partial \gamma^2} + L_{28}P_s;\\
&\frac{\partial \tau_{t\gamma}}{\partial t} = L_{31}\sigma_t + L_{32}\frac{\partial u_s}{\partial s} + L_{33}\frac{\partial^2 u_\gamma}{\partial s^2} + L_{34}\frac{\partial \sigma_t}{\partial \gamma} + L_{35}\frac{\partial^2 u_s}{\partial s \partial \gamma} + L_{36}\frac{\partial u_\gamma}{\partial \gamma} + L_{37}\frac{\partial^2 u_\gamma}{\partial \gamma^2} + L_{38}P_\gamma;\\
&\frac{\partial u_s}{\partial t} = L_{41}\tau_{st} + L_{42}\frac{\partial u_t}{\partial s};\\
&\frac{\partial u_t}{\partial t} = L_{51}\sigma_t + L_{52}\frac{\partial u_s}{\partial s} + L_{53}u_\gamma + L_{54}\frac{\partial u_\gamma}{\partial \gamma};\\
&\frac{\partial u_\gamma}{\partial t} = L_{61}\tau_{t\gamma} + L_{62}u_t + L_{63}\frac{\partial u_t}{\partial \gamma}.
\end{aligned}
\tag{17}
$$

Here

$$
\begin{aligned}
&L_{11} = -H_2; \quad L_{12} = -2\frac{\partial H_2}{\partial \gamma}; \quad L_{13} = -H_2; \quad L_{14} = -H_2;\\
&L_{21} = -H_2 M_{11}; \quad L_{22} = -H_2 M_{21}; \quad L_{23} = -\frac{\partial}{\partial \gamma}(H_2 M_{31});\\
&L_{28} = -H_2;\\
&L_{31} = \frac{\partial H_2}{\partial \gamma} - \frac{\partial}{\partial \gamma}(H_2 M_{21}); \quad L_{32} = \frac{\partial}{\partial \gamma}(H_2 M_{22}); \quad L_{33} = -H_2 M_{31};\\
&L_{34} = -H_2 M_{21}; \quad L_{35} = -H_2(M_{22} + M_{31}); \quad L_{36} = -\frac{\partial}{\partial \gamma}(H_2 M_{23});\\
&L_{37} = -H_2 M_{23}; \quad L_{38} = -H_2.
\end{aligned}
\tag{18}
$$

For dynamical problems, the system takes the form:

$$
\begin{aligned}
&\frac{\partial \sigma_t}{\partial t} = L_{11}\frac{\partial \tau_{st}}{\partial s} + L_{12}\tau_{t\gamma} + L_{13}\frac{\partial \tau_{t\gamma}}{\partial \gamma} + H_2\rho\frac{\partial^2 u_t}{\partial t_*^2};\\
&\frac{\partial \tau_{st}}{\partial t} = L_{21}\frac{\partial \sigma_t}{\partial s} + L_{22}\frac{\partial^2 u_s}{\partial s^2} + L_{23}\frac{\partial u_\gamma}{\partial s} + L_{24}\frac{\partial u_s}{\partial \gamma} + L_{25}\frac{\partial^2 u_\gamma}{\partial s \partial \gamma} + L_{26}\frac{\partial^2 u_s}{\partial \gamma^2} + H_2\rho\frac{\partial^2 u_s}{\partial t_*^2};\\
&\frac{\partial \tau_{t\gamma}}{\partial t} = L_{31}\sigma_t + L_{32}\frac{\partial u_s}{\partial s} + L_{33}\frac{\partial^2 u_\gamma}{\partial s^2} + L_{34}\frac{\partial \sigma_t}{\partial \gamma} + L_{35}\frac{\partial^2 u_s}{\partial s \partial \gamma} + L_{36}\frac{\partial u_\gamma}{\partial \gamma} + L_{37}\frac{\partial^2 u_\gamma}{\partial \gamma^2} + H_2\rho\frac{\partial^2 u_\gamma}{\partial t_*^2};\\
&\frac{\partial u_s}{\partial t} = L_{41}\tau_{st} + L_{42}\frac{\partial u_t}{\partial s};\\
&\frac{\partial u_t}{\partial t} = L_{51}\sigma_t + L_{52}\frac{\partial u_s}{\partial s} + L_{53}u_\gamma + L_{54}\frac{\partial u_\gamma}{\partial \gamma};\\
&\frac{\partial u_\gamma}{\partial t} = L_{61}\tau_{t\gamma} + L_{62}u_t + L_{63}\frac{\partial u_t}{\partial \gamma}.
\end{aligned}
\tag{19}
$$

Here

$$
\begin{aligned}
&L_{41} = \frac{H_2}{c_{66}}; \quad L_{42} = -H_2;\\
&L_{51} = \frac{H_2}{c_{22}}; \quad L_{52} = -\frac{H_2 c_{21}}{c_{22}}; \quad L_{53} = -\frac{\partial H_2}{\partial \gamma};\\
&L_{54} = -\frac{H_2}{c_{22}}c_{23}; \quad L_{55} = \frac{H_2}{c_{22}}g_2;\\
&L_{61} = \frac{H_2}{c_{44}}; \quad L_{63} = -H_2.
\end{aligned}
\tag{20}
$$

Next let us state a boundary-value problem for noncircular hollow cylinder, whose stress–strain state is described by the system of partial differential equations (17).

Consider noncircular hollow cylinder that is closed over the directrix.

In the general case when the symmetry of a stress–strain state is absent, the solution for closed cylinders at the expense of the satisfaction of the conditions of periodicity is reduced to that for nonclosed ones.

To reduce the dimensionality of the system (17), the given and sought-for functions are represented as Fourier series in the coordinate s:

$$X(s,t,\gamma) = \sum_{m=0}^{\infty} X_m(t,\gamma)\sin\lambda_m s,$$
$$Y(s,t,\gamma) = \sum_{m=0}^{\infty} Y_m(t,\gamma)\cos\lambda_m s, \tag{21}$$

where

$$X(s,t,\gamma) = \{\sigma_s,\sigma_t,\sigma_\gamma,\tau_{t\gamma},u_t,u_\gamma,e_s,e_t,e_\gamma,e_{t\gamma},p_t,p_\gamma,q_t,q_\gamma\},$$

$$Y(s,t,\gamma) = \{\tau_{st},\tau_{\gamma s},u_s,e_{st},e_{\gamma s},p_s,q_s\},$$

$$X_m(t,\gamma) = \{\sigma_{sm},\sigma_{tm},\sigma_{\gamma m},\tau_{t\gamma m},u_{tm},u_{\gamma m},e_{sm},e_{tm},e_{\gamma m},e_{t\gamma m},p_{tm},p_{\gamma m},q_{tm},q_{\gamma m}\},$$

$$Y_m(t,\gamma) = \{\tau_{stm},\tau_{\gamma sm},u_{sm},e_{stm},e_{\gamma sm},p_{sm},q_{sm}\},$$

where $\lambda_m = m\pi/L$.

While studying the wave processes occurring in anisotropic noncircular hollow cylinders, in order to decrease the dimensionality of the system of partial differential equations (19), we will present the solution as a traveling wave along the axis s in the form

$$\overline{V} = \{\sigma_t,\tau_{ts},\tau_{t\gamma},u_s,u_t,u_\gamma\} = \{\tilde{\sigma}_t(t,\gamma),\mathrm{i}\tilde{\tau}_{ts}(t,\gamma),\tilde{\tau}_{t\gamma}(t,\gamma),\tilde{\mathrm{i}}u_s(t,\gamma),\tilde{u}_t(t,\gamma),\tilde{u}_\gamma(t,\gamma)\}\mathrm{expi}(k_s s - \omega t_x^*), \tag{22}$$

where k_s is the wave number in the axial direction, ω is the cyclic frequency, and $t*$ is the temporal coordinate. Below, we will omit the sign "tilde".

Applying expansions (21) and (22) to the three-dimensional problems (17) and (19), we obtain a number of two-dimensional problems for every harmonic of the expansion. The two-dimensional boundary-value problem is reduced to one-dimensional ones by employing the method of straight lines over the thickness of the cylinder. All the coefficients of the system of equations (17) and (19), as well as the solving functions, are assumed to be sufficiently smooth in the coordinate γ. Now, the derivatives in this coordinate are replaced by finite-difference relations, and the real boundary conditions at cross-sections $\theta = \text{const.}$ are substituted by their discrete values at equidistant surfaces.

In this paper, a more accurate variant of approximation is used that makes it possible to reduce the error to $O(h^n)$, where n is the number of points of approximation across the thickness of the cylinder. We call it as a general approximation. The efficiency of various variants of approximation is demonstrated below by several examples.

Thus, having applied the method of straight lines to a two-dimensional problem formulated for every mth harmonic of the expansion we arrive to the following system of k ordinary differential equations ($k = 6n - 4$, where n is the number of points of finite-difference approximation across the thickness of the cylinder):

$$\frac{\mathrm{d}\overline{V}}{\mathrm{d}t} = A(t)\overline{V}(t) + \overline{B}(t), \tag{23}$$

$$A_1\overline{V}(t_0) = \overline{B_1},$$
$$A_2\overline{V}(t_N) = \overline{B}_2, \tag{24}$$

where

$$\overline{V} = \left\{ \begin{matrix} \sigma_t^1,\tau_{st}^1,u_s^1,u_t^1,\sigma_t^2,\tau_{st}^2,\tau_{t\gamma}^2,u_s^2,u_t^2,u_\gamma^2,\ldots, \\ \sigma_t^{n-1},\tau_{st}^{n-1},\tau_{t\gamma}^{n-1},u_s^{n-1},u_t^{n-1},u_\gamma^{n-1},\sigma_t^n,\tau_{st}^n,u_s^n,u_t^n \end{matrix} \right\}$$

is the solving vector function; $A(t)$ is the $N^* \times N^*$ order matrix of coefficients of the system; A_1 and A_2 are the $\left(\frac{N^*}{2} \times N^*\right)$ order rectangular matrices and $\overline{B}(t), \overline{B}_1, \overline{B}_2$ are the given vectors with the dimensionless values $N^*, \frac{N^*}{2}, \frac{N^*}{2}$.

By studying the stationary dynamic processes in anisotropic hollow noncircular cylinders, we arrive to the generalized problem on eigenvalues for systems of linear ordinary differential equations with relevant boundary conditions:

$$\frac{\partial \overline{V}}{\partial t} = D(t, k_s, \omega); \tag{25}$$

$$D_1 \overline{V}(t_0) = 0; \tag{26}$$

$$D_2 \overline{V}(t_n) = 0. \tag{27}$$

Here, $D(t, k_s, \omega)$ is the matrix of coefficients of the $N^* \times N^*$ order system and D_1, D_2 are the rectangular matrixes of the $\frac{N^*}{2} \times N^*$ order.

To solve the linear boundary-value problem described above we will use the stable numerical method of discrete orthogonalization (Grigorenko et al., 1986), whereas for solution of the generalized problem on eigenvalues we will employ the method of discrete orthogonalization in combination with the method of stepwise search.

4. Analysis of a stress–strain state of cylinders with ellipsoidal cross-section

Let us demonstrate the advantages of the suggested approach by solving the stress–strain problem for a thick-walled cylinder with an arbitrary cross-section. Having verified the realizability of the approach by solving the problem on deformation of a circular thick-walled cylinder that has the exact solution, we can solve the stress–strain problems for the thick-walled cylinder with a more complex geometry of cross-section.

As an example, we will consider the problem on the stress–stain state of a noncircular hollow cylinder, whose cross-section contour has the form of an ellipse (Fig. 2). The corresponding equations in the parametric form can be written as follows:

$$x = b \cos \theta, \quad z = a \sin \theta,$$

where a and b are the ellipse semi-axes, and θ is a some parameter.

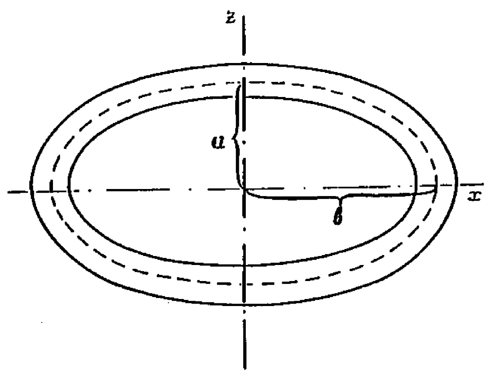

Fig. 2. Cross-section of the cylinder under consideration.

Assume that the perimeter of the cylinder cross-section contour remains unchangeable and equal to the length of the circumference with the radius R. In this case, we have the equality

$$\pi(a+b)\left(1+\frac{\Delta^2}{4}+\frac{\Delta^4}{64}+\frac{\Delta^6}{256}+\cdots\right)=2\pi R.$$

Supposing that $\Delta=\frac{b-a}{b+a}$ and keeping the terms up to Δ^6, we obtain:

$$a=\frac{R}{f}(1-\Delta),\quad b=\frac{R}{f}(1+\Delta);$$

$$f=1+\frac{\Delta^2}{4}+\frac{\Delta^4}{64}+\frac{\Delta^6}{256},\quad \frac{a}{b}=\frac{1-\Delta}{1+\Delta}.$$

By varying Δ, we can change the ratio of the ellipse semi-axes. So, in particular, at $\Delta=0$, we have a circular cylinder.

Having chosen the parameter θ as the variable of integration, we will introduce the new factor β, which is related to θ as follows:

$$\beta=\beta(\theta)=\sqrt{b^2\sin^2\theta+a^2\cos^2\theta}.$$

Then the directrix curvature of an ellipsoidal cylinder is defined as

$$K_t=\frac{ab}{\sqrt{(b^2\sin^2\theta+a^2\cos^2\theta)^3}}.$$

At first, we will study the stress–strain state of an isotropic hollow cylinder for various values of geometric characteristics. Let the cylinder is under the action of the internal pressure $q_\gamma=q_0\sin\frac{\pi s}{\ell}\,(q_0=\text{const.})$.

By virtue of the symmetry of the stress–strain state relative to the planes $\theta=0$ and $\pi/2$, it is sufficient to consider the quarter of the cylinder bounded by these planes, having specified the boundary conditions in the sections $\theta=0$ and $\theta=\pi/2$ as follows:

$$\tau^i_{st}=0,$$
$$\tau^i_{t\gamma}=0,$$
$$u^i_t=0.$$

In calculations we adopt: $E=E_0$; $\nu=0.3$; $R=50$; $\ell=50$, 100, 200; $H=5$, 10, 15, 20; and $\Delta=0$, 0.1, 0.2, 0.3.

Let us estimate the influence of variation in the parameter Δ, which characterizes the degree of deviation of the shape of the cross-section from the circular one, on the stress–strain state of the thick-walled cylinder with an ellipsoidal cross-section. The calculations were performed for the following values of the geometric parameters: $R=50$; $\ell=50$; $H=10$; and $\Delta=0$, 0.1, 0.2, 0.3.

The graphs of distributions of the longitudinal σ_s and circumferential σ_t stresses in the domain $\left(0\leqslant\theta\leqslant\frac{\pi}{2}\right)$ on the inner ($\gamma=-\frac{H}{2}$; solid line) and outer ($\gamma=\frac{H}{2}$; dashed line) surfaces of the cylinder and deflections u_γ of the middle surface for various values of Δ along the directrix are presented in Figs. 3–5. As it is seen from Figs. 3 and 4, the longitudinal σ_s and circumferential σ_t stresses in the circular cylinder are constant along the total length of the directrix. But they change significantly with increase in Δ in the case of the noncircular cylinder. From the graphs presented it follows that the character of the stress state on the inner surface of the cylinder in the zone of minimum rigidity ($\theta=\pi/2$) is defined, beginning from $\Delta=0.1$, by the longitudinal stresses σ_s that exceed significantly the circumferential stresses σ_t. In contrast, in the zone of maximum rigidity ($\theta=0$), the circumferential stresses σ_t dominate. For the outer surface the values

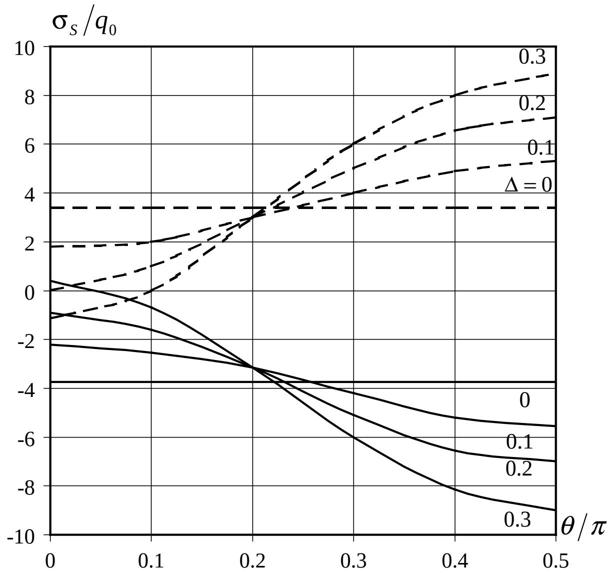

Fig. 3. Distribution of the longitudinal stresses on the inner (solid line) and outer (dashed line) surfaces along the cylinder directrix for various values of the ellipticity parameter.

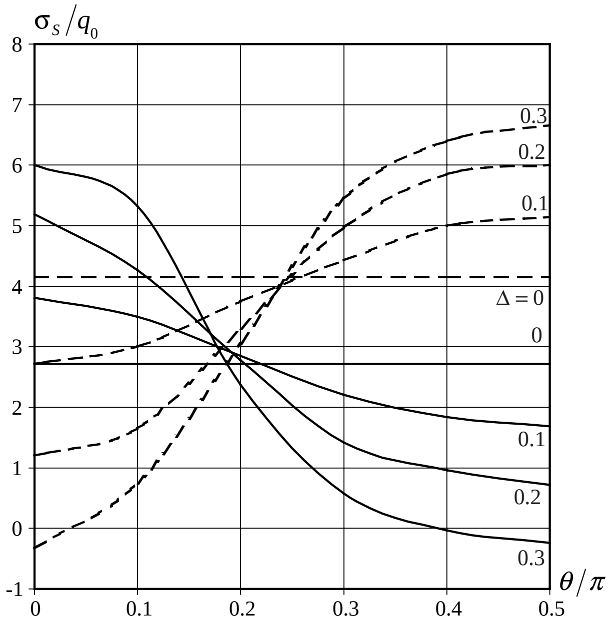

Fig. 4. Distribution of the circumferential stresses on the inner (solid line) and outer (dashed line) surfaces along the cylinder directrix for various values of the ellipticity parameter.

of longitudinal and circumferential stresses in the mentioned zones are close. The longitudinal stresses σ_s both on the inner and outer surfaces reach their maximum in the zone of minimum rigidity of the cylinder for all of the values Δ. In some section $\theta = \text{const.}$ for all the values of the parameter Δ, the longitudinal σ_s

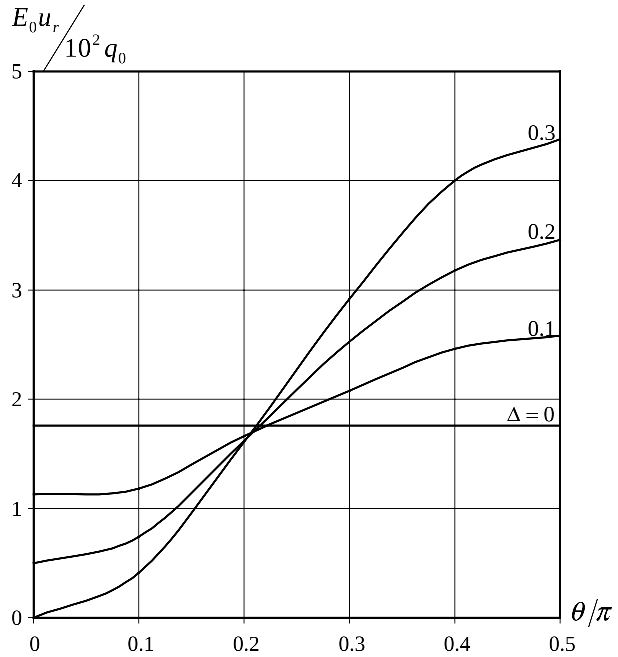

Fig. 5. Distribution of the deflection of the middle surface along the cylinder directrix for various values of the ellipticity parameter.

and circumferential stresses σ_t are close to magnitudes of appropriate stresses in a circular cylinder ($\Delta = 0$). Such a phenomenon for the longitudinal stresses can be seen in the section $\theta = \pi/6$ both on the inner and outer surfaces. As for circumferential stresses, it should be noted that their values for circular and noncircular cylinders coincide also in the cross-section $\theta = \pi/6$.

The distribution of deflections u_γ along the directrix of the middle surface is shown in Fig. 5. Note that the nonlinearity of the distribution increases with Δ.

The analysis of deflections of the middle surface indicates to their increase in the zone of minimum rigidity with change in Δ and their reduction in the zone of maximum rigidity. Such relations are typical for cylinders with an ellipsoidal cross-section. In the section $\theta = \pi/6$, the magnitudes of deflections of the middle surface for all the values of Δ are close to those which are held in a circular cylinder.

One of the advantages of the suggested method is its three-dimensionality, i.e., the ability to present the distribution of the parameters of a stress–strain state along three directions. Of peculiar interest is the distribution of the stress state factors over the cylinder thickness.

The graphs of the distribution of the circumferential stresses σ_t over the cylinder thickness in the zones of maximum (dashed line) and minimum (solid line) rigidities for all of the values of Δ are presented in Fig. 6.

The distinctive feature of the distribution of σ_t over the thickness is the increase in nonlinearity in the zone of maximum rigidity with increase in the parameter Δ. The behaviour of the distribution σ_t in the zone of minimum rigidity is close to a linear one. The high values of σ_t are achieved at the loaded surface in the zone of maximum rigidity and at the unloaded one in the zone of minimum rigidity. The values of σ_t at a surface close to the middle one in noncircular cylinders are close to those in a circular one.

The values of deflections u_γ distributed over the cylinder thickness in the zones of minimum and maximum rigidities for all the parameters Δ are presented in Table 1. The analysis of numerical values testifies that the increase in the values of deflections with a growth of Δ in the zone of minimum rigidity and their reduction down to a change of the sign ($\Delta = 0.3$) in the zone of maximum rigidity is characteristic for

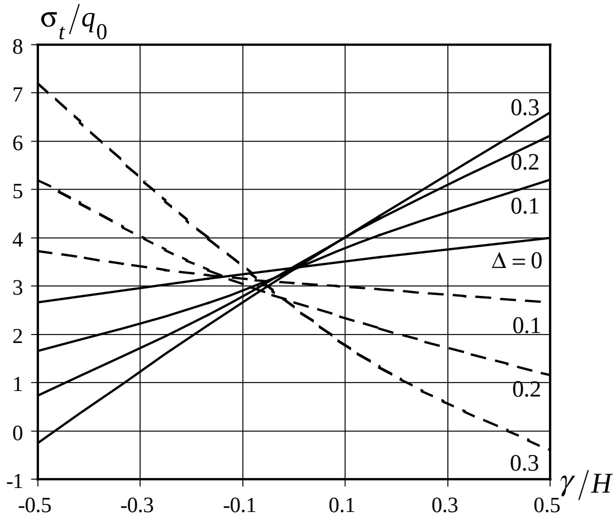

Fig. 6. Distribution of the circumferential stresses over the cylinder thickness in the zones of maximal (dashed line) and minimal (solid line) rigidity for various values of the ellipticity parameter.

Table 1
Distribution of deflection over the cylinder thickness

θ	γ/H	$u_\gamma/E_0^{-1}q_0$			
		$\Delta = 0$	$\Delta = 0.1$	$\Delta = 0.2$	$\Delta = 0.3$
0	−1/2	182.43	110.76	52.145	7.8457
	−1/3	181.04	108.39	48.872	3.7254
	−1/6	179.18	106.17	46.471	1.4345
	0	176.92	104.04	44.652	0.21605
	1/6	174.29	101.94	43.250	−0.29282
	1/3	171.26	99.845	42.180	−0.25600
	1/2	167.78	97.705	41.392	0.26557
$\pi/2$	−1/2	182.43	263.89	349.98	434.42
	−1/3	181.04	263.54	350.73	436.28
	−1/6	179.18	262.19	349.99	436.22
	0	176.92	259.97	347.94	434.45
	1/6	174.29	256.93	344.65	431.04
	1/3	171.26	253.08	340.12	425.99
	1/2	167.78	248.34	334.24	419.17

cylinders with an ellipsoidal cross-section. For a slight deviation of the shape of a cross-section from the circular one, i.e., for $\Delta \leqslant 0.2$, the character of the distribution of the values of deflections over the thickness is close to the linear one both in the zones of minimum and maximum rigidities of the cylinder. But for $\Delta > 0.2$, i.e., in the case of a considerable deviation of the shape of the cross-section from the circular one, the distribution of deflections in the zones of minimum and maximum rigidities is nonlinear.

5. Solution of the problem on propagation of elastic waves in a cylinder with ellipsoidal cross-section

Let us consider the problem on propagation of elastic waves in a hollow noncircular orthotropic cylindrical waveguide with an ellipsoidal cross-section. The geometry of noncircular hollow cylinders with such a cross-section was described in the previous section. Due to the symmetry of a stress–strain state relative to the planes $t = 0$ and $t = \pi/2$, it is sufficient to consider the half of a noncircular cylinder, that is bounded by these planes, by setting the boundary conditions at the cross-sections $t = 0$ and $t = \pi/2$ as follows:

$$\sigma_{st}^i = 0 \quad (i = 1, 2, \ldots, n),$$
$$\sigma_{t\gamma}^i = 0 \quad (i = 1, 2, \ldots, n),$$
$$u_t^i = 0 \quad (i = 2, 3, \ldots, n-1).$$

As an example of application of the proposed approach, we will consider the problem on propagation of elastic waves in the waveguide with thickness $H = 4$. The perimeter of the middle surface contour of its cross-section is equal to the length of the circumference with the radius $R = 2$. The elastic characteristics of the orthotropic material have the following values:

$$20E_1 = E_3, \quad E_2 = E_3, \quad 20G_{12} = 20G_{13} = E_3, \quad 2.68G_{23} = E_3, \quad \nu_{12} = \nu_{23} = 0.34; \quad \nu_{13} = 0.017.$$

To estimate the accuracy of the proposed approach, the results of calculation of the dynamical characteristics for a noncircular orthotropic cylinder are compared with the data that were derived within the method developed for circular cylinders. The values of the first six dimensionless frequencies $\omega^* = \omega H \sqrt{\rho/G_{12}}$ at a fixed wave number that were derived within various methods are given in Table 2. To solve the dynamic problems, we carried out the calculations by the proposed approach in the case of a noncircular cylinder for following numbers of points of the difference approximation over the coordinate related to thickness: $n = 7$, $n = 9$, and $n = 11$. The practical coincidence of the values of frequencies, which were found within the two different approaches (the difference ranges from 0.08% to 2.12% and decreases with increase in the number of straight lines), makes it possible to estimate the accuracy of the results derived in this section.

For the isoperimetric noncircular cylinder under study, we determined the dispersion curves for various values of the parameter of ellipticity Δ. The values of the first cut-off frequencies upon growing Δ ($\Delta = 0.1$–0.6) are given in Table 3. In this case, having compared the cut-off frequencies with those for a circular cylinder ($R = 2, H = 4$), we observe a slight decrease of these frequencies with increase in ellipticity (for various cut-off frequencies, the difference is within the range 1.5–36%). The behaviour of the dispersion curves ($\omega^* = \omega H \sqrt{\rho/G_{12}}$) for circular and noncircular orthotropic cylinders at the first and fourth frequencies is compared in Fig. 7. Here the solid lines denote the dispersion curves for a circular cylinder, dashed ($\Delta = 0.5$) and dash-dotted ($\Delta = 0.2$) lines correspond to those for noncircular cylinders. Due to this date one can conclude that the difference in the behaviour of dispersion curves becomes more pronounced with increase in the wave number (at $\kappa^* = 0.4$, it equals 10% at $\Delta = 0.2$% and 17.2% at $\Delta = 0.5$ for the first mode as well as 2.8% at $\Delta = 0.2$% and 7.2% at $\Delta = 0.5$ for the fourth mode). It should be noted also that the values of frequencies decrease with increase in the ellipticity parameter Δ. For elliptic cross-sections, the first modes also propagate practically without dispersion.

Table 2
Comparison of the first six frequencies, obtained using different approaches, for specific wave numbers

No.	According to the circular cylinder method	$n = 7$	$n = 9$	$n = 11$
1	0.5581	0.5649	0.5608	0.5597
2	0.5600	0.5651	0.5613	0.5603
3	0.5731	0.5771	0.5741	0.5734
4	0.5794	0.5842	0.5817	0.5814
5	0.5806	0.5874	0.5826	0.5821
6	0.6219	0.6296	0.6243	0.6218

Table 3
Values of the cut-off frequencies for different ellipticity parameters

No.	Δ			
	0	0.1	0.2	0.3
2	0.0892	0.0878	0.0859	0.0826
3	0.1938	0.1927	0.1908	0.1819
4	0.2492	0.2478	0.2456	0.2448
5	0.2741	0.2732	0.2724	0.2714
	0	0.4	0.5	0.6
2	0.0892	0.0778	0.0719	0.0662
3	0.1938	0.1771	0.1683	0.1622
4	0.2492	0.2436	0.2427	0.2419
5	0.2741	0.2702	0.2685	0.2619

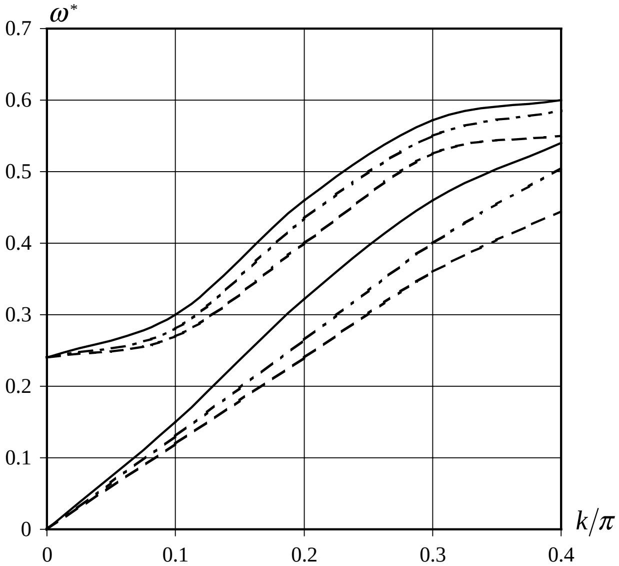

Fig. 7. Dispersion curves for noncircular orthotropic cylinders at various values of the parameter ellipticity parameter: solid lines correspond to the circular cylinder, dashed and dot-dashed lines correspond to the noncircular cylinder for $\Delta = 0.5$ and $\Delta = 0.1$, respectively.

6. Conclusion

In this paper, we have proposed the numerical–analytical approach for analyzing the stress–strain state and character of propagation of harmonic elastic waves in an thick-walled anisotropic hollow cylinder with a noncircular cross-section. The approach consists of three stages. At the first stage the three-dimensional problem is reduced to the two-dimensional one with the solution being presented as the Fourier series expansion in the longitudinal coordinate. At the second stage the solution is approximated by finite differences over the cylinder thickness that is assumed essential in comparison with other dimensions. At the third stage the one-dimensional boundary-value problems and eigenvalue problem are solved by the numerical method of discrete orthogonalization. In the case of a dynamic problem this method is combined with the method of a step-by-step search.

It should be noted that the approach used at the first and third stages is realized with sufficient accuracy, whereas at the second stage its error may be estimated. Since the body under consideration has not inclusions or cut-outs, the approach needs to be recognized as more efficient when compared with the finite element method.

As an example, the cylinder with an elliptical cross-section was considered. The results of the investigation into the stress–strain state and dispersion curves are presented depending on the cross-section ellipticity parameters.

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